

# Entropy method for the left tail

Hyungsu Kim,<sup>1,a</sup> Chul Ki Ko,<sup>2</sup> and Sungchul Lee<sup>3,a</sup>

## Abstract

When we use the entropy method to get the tail bounds, typically the left tail bounds are not good comparing with the right ones. Up to now this asymmetry has been observed many times. Surprisingly we find an entropy method for the left tail that works in the resembling way that it works for the right tail. This new method does not work in all the cases. We provide a meaningful example.

## 1 Introduction.

In recent years, interesting developments took place in the analysis of the spectrum of large random matrices. In particular, the asymptotic distribution of the largest eigenvalue has been a subject of hot interest.

Let  $\mathbf{X} = (X_{ij})$  be an  $n \times n$  complex hermitian matrix such that the entries  $X_{ij}$  on and above diagonal are independent complex (real on the diagonal) centered normal random variables with variance 1. Let  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$  be the  $n$  real eigenvalues of  $\frac{1}{\sqrt{n}}\mathbf{X}$ . There have been many researches of the concentration of the largest eigenvalue  $\lambda_1$  or the concentration of the  $k$ -th largest eigenvalue  $\lambda_k$ . Regarding the concentration of the  $k$ -th largest eigenvalue, we know of three results; Alon, Krivelevich, Vu (2002), Meckes (2004), Maurer (2006). Alon, Krivelevich, Vu (2002) and Meckes (2004) used Talagrand's method whereas Maurer (2006) used the entropy method. Since our main theme of this paper is the entropy method, we state Maurer's concentration result.

**Theorem.**[Maurer (2006)] Let  $\mathbf{X} = (X_{ij})$  be an  $n \times n$  real symmetric matrix such that the entries  $X_{ij}$  on and above diagonal are independent with  $|X_{ij}| \leq 1$ . Let  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$  be the  $n$  real eigenvalues of  $\mathbf{X}$ . Then, for all  $k$ ,  $n \geq 1$ , and for all  $t \geq 0$ ,

$$P(\lambda_k - E\lambda_k \geq t) \leq \exp\left(-\frac{t^2}{16k^2}\right), \quad P(\lambda_k - E\lambda_k \leq -t) \leq \exp\left(-\frac{t^2}{16k^2 + 2kt}\right). \quad (1.1)$$

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<sup>1</sup>Department of Mathematics, Yonsei University, Seoul, Korea 120-749.

E-mail: gudtn@yonsei.ac.kr

<sup>2</sup>University College, Yonsei University, Seoul, Korea 120-749. E-mail: kochulki@yonsei.ac.kr

<sup>3</sup>Department of Mathematics, Yonsei University, Seoul, Korea 120-749.

E-mail: sungchul@yonsei.ac.kr.

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The left tail bounds in (1.1) are larger than the right ones. This asymmetry usually happens when we use the entropy method to get the tail bounds. However, this asymmetry is not observed in the works of Alon, Krivelevich, Vu (2002) and Meckes (2004) which are based on Talagrand's method. In these works the left tail bounds are same to the right ones. In addition, the centering is the median not the mean. This symmetry and the centering are typical with Talagrand's method.

In this paper we found an entropy method for the left tail that works in the resembling way that it works for the right tail by controlling the term  $\Delta^2$  carefully (see (2.9) in Section 2 for the definition of  $\Delta^2$ ), and give a meaningful example.

The rest of the paper is organized as follows. In Section 2, we develop an entropy method for the left tail. In Section 3, we apply this new method to the interesting case including the  $k$ -th largest eigenvalue.

## 2 Entropy method for the left tail.

The concentration of measure phenomenon for the product measures has been investigated in depth by Talagrand (1995, 1996) in a most remarkable way. His method has been applied to various interesting cases. In many cases his method made new-record concentration inequalities and in some cases his method even produced non-trivial concentration inequalities for the first time. However, his method is technically too complicated. Hence many people tried to simplify his proof and studied to find an alternative to reproduce and more ambitiously to extend his result. One of the successful alternatives is the entropy method. Here we explain the minimum details of the entropy method to show our contributions on this interesting subject. See Ledoux (1996), Massart (2000), Boucheron, Lugosi, Massart (2000, 2003), Maurer (2006) for the full details.

Let  $X_1, \dots, X_n$  be independent and let  $G = G(X_1, \dots, X_n) > 0$ . Define the entropy  $H(G)$  and the partial entropy  $H_k(G)$  by

$$\begin{aligned} H(G) &:= EG \log G - EG \log EG, \\ H_k(G) &:= E_k G \log G - E_k G \log E_k G, \end{aligned}$$

where  $E$  is the integration over  $X_1, \dots, X_n$  whereas  $E_k$  is the integration over  $X_k$  only. So, the entropy  $H(G)$  is a real number but the partial entropy  $H_k(G)$  is a random variable which does not depends on  $X_k$ .

Some classical formulas of the entropy are quite helpful;

$$H(G) = \sup_T EG(\log T - \log ET), \quad (2.1)$$

$$H(G) = \inf_c EG(\log G - \log c) - (G - c), \quad (2.2)$$

where the supremum in (2.1) is taken over the strictly positive random variables  $T$  and where the infimum in (2.2) runs over the strictly positive constants  $c$ . (2.1) is called the duality formula of the entropy and (2.2) is called the variation formula.

Here is the well-known entropy inequality (or tensorization inequality) which follows from the duality formula (2.1).

**Lemma 1.** [Entropy inequality]

$$H(G) \leq \sum_{k=1}^n EH_k(G). \quad (2.3)$$

Now, let  $Z = Z(X_1, \dots, X_n)$  be the random variable of interest. We apply the entropy inequality to the random variable  $e^{\lambda Z}$ . Then, we have

$$E\lambda Ze^{\lambda Z} - Ee^{\lambda Z} \log Ee^{\lambda Z} \leq \sum_{k=1}^n EH_k(e^{\lambda Z}). \quad (2.4)$$

To estimate the term  $EH_k(e^{\lambda Z})$ , we apply the variation formula (2.2) to the partial entropy  $H_k(e^{\lambda Z})$ ;  $H_k(e^{\lambda Z}) = \inf_c E_k e^{\lambda Z} (\lambda Z - \log c) - (e^{\lambda Z} - c)$ . Since the integration  $E_k$  is only over  $X_k$ , during the evaluation of the partial entropy  $H_k(e^{\lambda Z})$  we can treat all the other random variables  $X_j$ ,  $1 \leq j \neq k \leq n$ , as fixed constants. So, in fact  $c$  can be chosen as a function of  $X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n$ , or even as a function of  $X_1, \dots, X_{k-1}, X'_k, X_{k+1}, \dots, X_n$ , where  $X'_k$  is an independent copy of  $X_k$  and  $X'_k$  is independent to  $X_1, \dots, X_n$ . This subtle point on  $c$  is crucial for the further development of the theory. If we choose a particular “constant”  $c_0$  to estimate the partial entropy  $H_k(e^{\lambda Z})$ , then we have

$$H_k(e^{\lambda Z}) \leq E_k e^{\lambda Z} (\lambda Z - \log c_0) - (e^{\lambda Z} - c_0). \quad (2.5)$$

To get a good concentration inequality, we have to choose  $c_0$  well-designed for the random variable  $Z$  of interest.

There are many possible choices of  $c_0$ . Massart (2000) and Boucheron, Lugosi, Massart (2003) chose

$$c_0 := \exp(\lambda Z(X_1, \dots, X'_k, \dots, X_n)) := e^{\lambda Z_k},$$

where  $X'_k$  is an independent copy of  $X_k$ .

Boucheron, Lugosi, Massart (2000) chose

$$c_0 := \exp \left( \lambda Z(X_1, \dots, \hat{X}_k, \dots, X_n) \right) := e^{\lambda Z_k}.$$

Here,  $\hat{X}_k$  means that we drop out  $X_k$  from the argument of  $Z$ . In other word, we evaluate the value of  $Z$  based not  $\{X_1, \dots, X_n\}$  but  $\{X_1, \dots, X_n\} \setminus \{X_k\}$ . This is possible because of the special nature of the random variable  $Z$  they considered.

Maurer (2006) chose

$$c_0 := \exp \left( \lambda \inf_{x_k} Z(X_1, \dots, x_k, \dots, X_n) \right) := e^{\lambda Z_k}, \quad (2.6)$$

where the infimum runs over all the possible values  $x_k$  which  $X_k$  can take as a function value or over a compact set containing the support of the distribution of  $X_k$ . He used this  $c_0$  (or  $Z_k$ ) to get the right tail bound in Theorem A. He also use the same  $Z_k$  to obtain the left tail bound in the same Theorem.

In this paper we follow the footsteps of Maurer for the right tail bound. However, to get a better left tail bound we choose the following  $c_0 = e^{\lambda Z_k}$  for the left tail bound;

$$c_0 := \exp \left( \lambda \sup_{x_k} Z(X_1, \dots, x_k, \dots, X_n) \right) := e^{\lambda Z_k}, \quad (2.7)$$

where the supremum runs over a compact set containing the support of the distribution of  $X_k$ . This choice does not always come with a sensible  $\Delta^2$  (see (2.9) below for the definition of  $\Delta^2$ ). However, in many cases with this choice we do have  $\Delta^2$  with  $\|\Delta^2\|_\infty < \infty$ .

Let's recall what we have done so far with the entropy inequality. We first apply the entropy inequality to  $G = e^{\lambda Z}$  where  $Z$  is the random variable of interest. Then, the term  $EH_k(e^{\lambda Z})$  appears in the inequality. To estimate the term  $EH_k(e^{\lambda Z})$ , with a particular choice  $c_0 = e^{\lambda Z_k}$  we apply the variation formula to  $H_k(e^{\lambda Z})$ . Then, we get the following log-Sobolev inequality.

**Lemma 2.** [Log-Sobolev inequality] If  $-\lambda(Z - Z_k) \leq 0$  for all  $k$ , then

$$E\lambda Z e^{\lambda Z} - Ee^{\lambda Z} \log Ee^{\lambda Z} \leq \frac{\lambda^2}{2} Ee^{\lambda Z} \Delta^2, \quad (2.8)$$

where

$$\Delta^2 := \sum_{k=1}^n (Z - Z_k)^2. \quad (2.9)$$

**Proof.** With a particular choice  $c_0 = e^{\lambda Z_k}$ , from (2.5) we have

$$\begin{aligned} H_k(e^{\lambda Z}) &\leq E_k e^{\lambda Z} (e^{-\lambda(Z-Z_k)} - (1 - \lambda(Z - Z_k))) \\ &= E_k e^{\lambda Z} \frac{e^{-\lambda(Z-Z_k)} - (1 - \lambda(Z - Z_k))}{\lambda^2(Z - Z_k)^2} \lambda^2(Z - Z_k)^2. \end{aligned}$$

If  $-\lambda(Z - Z_k) \leq 0$  for all  $k$ , since  $(e^x - (1 + x))/x^2$  is an increasing function with the function value  $1/2$  at the trouble spot  $x = 0$ , and (hence) since  $(e^x - (1 + x))/x^2 \leq 1/2$  for  $x \leq 0$ , we have then

$$H_k(e^{\lambda Z}) \leq \frac{\lambda^2}{2} E_k e^{\lambda Z} (Z - Z_k)^2.$$

Plug this estimate into (2.4) and we get the log-Sobolev inequality (2.8). ■

To distinguish our choice (2.7) from Maurer's choice (2.6), from now on we let

$$\begin{aligned} \Delta_M^2 &:= \sum_{k=1}^n \left( Z - \inf_{x_k} Z(X_1, \dots, x_k, \dots, X_n) \right)^2 := \sum_{k=1}^n \left( Z - Z_k^{(M)} \right)^2, \\ \Delta_L^2 &:= \sum_{k=1}^n \left( Z - \sup_{x_k} Z(X_1, \dots, x_k, \dots, X_n) \right)^2 := \sum_{k=1}^n \left( Z - Z_k^{(L)} \right)^2. \end{aligned}$$

Here is our entropy method for the left tail, which is a simple consequence of the log-Sobolev inequality.

**Theorem 1.** (i) If  $\|\Delta_M^2\|_\infty \leq \infty$ , then for  $t \geq 0$

$$P(Z - EZ \geq t) \leq \exp \left( -\frac{t^2}{2\|\Delta_M^2\|_\infty} \right). \quad (2.10)$$

(ii) If  $\|\Delta_L^2\|_\infty \leq \infty$ , then for  $t \geq 0$

$$P(Z - EZ \leq -t) \leq \exp \left( -\frac{t^2}{2\|\Delta_L^2\|_\infty} \right). \quad (2.11)$$

**Remark.** As Maurer pointed out in private communication,  $\|\Delta_M^2\|_\infty \neq \|\Delta_L^2\|_\infty$ . However, in practice we don't know the exact values of  $\|\Delta_M^2\|_\infty$  and  $\|\Delta_L^2\|_\infty$ . Instead we calculate the upper bounds of  $\|\Delta_M^2\|_\infty$  and  $\|\Delta_L^2\|_\infty$ . In case  $\|\Delta_M^2\|_\infty = \|\Delta_L^2\|_\infty < \infty$ , (2.10) and (2.11) provide the same left and right tail bounds.

**Proof.** The right tail bound (2.10) is Theorem 1 of Maurer (2006). So, we can safely skip its proof. In fact, the left tail bound (2.11) also follows from the same

argument, the so-called Herbst's argument. For reader's convenience here we reproduce the Herbst's argument to get (2.11).

In this proof, we will use only the negative  $\lambda \leq 0$ . Then, (since by our choice of  $Z_k^{(L)}$ ,  $Z - Z_k^{(L)} \leq 0$ ) we have  $-\lambda(Z - Z_k) \leq 0$  for all  $k$ . So, we can use the log-Sobolev inequality (2.8). Since  $\|\Delta_L^2\|_\infty \leq \infty$ , by (2.8)

$$E\lambda Z e^{\lambda Z} - E e^{\lambda Z} \log E e^{\lambda Z} \leq \frac{\lambda^2}{2} \|\Delta_L^2\|_\infty E e^{\lambda Z}.$$

Divide the both sides by  $\lambda^2 E e^{\lambda Z}$ . Then, we have

$$\frac{d}{d\lambda} \frac{1}{\lambda} \log E e^{\lambda(Z-EZ)} \leq \frac{\|\Delta_L^2\|_\infty}{2}.$$

Recall  $\lambda \leq 0$ . So, we integrate the both sides from  $\lambda$  to 0. Since  $\lambda^{-1} \log E e^{\lambda(Z-EZ)} \rightarrow 0$  as  $\lambda \rightarrow 0$ , we have then  $-\lambda^{-1} \log E e^{\lambda(Z-EZ)} \leq -\|\Delta_L^2\|_\infty \lambda/2$  or

$$E e^{\lambda(Z-EZ)} \leq \exp\left(\frac{\|\Delta_L^2\|_\infty}{2} \lambda^2\right). \quad (2.12)$$

Now, by Chebyshev's inequality with the choice  $\lambda = -t/\|\Delta_L^2\|_\infty \leq 0$  we have the left tail bound (2.11); by (2.12),

$$P(Z - EZ \leq -t) \leq e^{\lambda t} E e^{\lambda(Z-EZ)} \leq \exp\left(\lambda t + \frac{\|\Delta_L^2\|_\infty}{2} \lambda^2\right) = \exp\left(-\frac{t^2}{2\|\Delta_L^2\|_\infty}\right).$$

■

### 3 Example.

In this section, we apply the entropy method for the left tail (Theorem 1) to the eigenvalues of sample covariance matrix. In a near future we hope to see many more exciting examples.

Let  $\mathbf{X} = (X_{ij})$  be an  $n \times N$  complex matrix with the independent entries  $X_{ij}$ . Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the  $n$  positive eigenvalues of  $\frac{1}{N} \mathbf{X} \mathbf{X}^*$ . Then, under the suitable condition on the distribution of  $X_{ij}$  the Marčenko-Pastur theorem (Marčenko and Pastur (1967)) says that as  $n \rightarrow \infty$ ,  $N \rightarrow \infty$ ,  $n/N \rightarrow c$  ( $0 < c < \infty$ ), the empirical spectral distribution  $\frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k}$  of the sample covariance matrix  $\frac{1}{N} \mathbf{X} \mathbf{X}^*$  converges to the Marčenko-Pastur law. This time we use the Marčenko-Pastur scaling. For the sample covariance matrix we don't know any established concentration inequality to compare with. So, it is rather natural to work with the Marčenko-Pastur scaling. Here

is our result.

**Theorem 2.** Let  $\mathbf{X} = (X_{ij})$  be an  $n \times N$  complex matrix with the independent entries  $X_{ij}$ , which are bounded by 1, i.e.,  $|X_{ij}| \leq 1$ . Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the  $n$  positive eigenvalues of  $\frac{1}{N}\mathbf{X}\mathbf{X}^*$ . Then, for all  $k, n, N \geq 1$ , and for all  $t \geq 0$ ,

$$P(\lambda_k - E\lambda_k \geq t) \leq \exp\left(-\frac{Nt^2}{2n^2}\right), \quad P(\lambda_k - E\lambda_k \leq -t) \leq \exp\left(-\frac{Nt^2}{2n^2}\right).$$

**Proof.** Let  $X_t$  be the  $t$ -th column of  $\mathbf{X}$ . To denote the dependency of the eigenvalues on the matrix  $\mathbf{X}$ , we let  $\lambda_1(\mathbf{X}) \geq \lambda_2(\mathbf{X}) \geq \dots \geq \lambda_n(\mathbf{X})$  be the  $n$  positive eigenvalues of  $\frac{1}{N}\mathbf{X}\mathbf{X}^*$ . Fix  $1 \leq k \leq n$  and let  $Z := Z(\mathbf{X}) := \lambda_k(\mathbf{X})$  be the  $k$ -th largest eigenvalue of  $\frac{1}{N}\mathbf{X}\mathbf{X}^*$ .

Fix  $1 \leq t_0 \leq N$ . From the given  $n \times N$  matrix  $\mathbf{X}$  delete the  $t_0$ -th column  $X_{t_0}$  and add  $x_{t_0}$  where  $x_{t_0}$  is a constant column vector of size  $n$  whose entries are all bounded by 1. Call this new  $n \times N$  matrix as  $\mathbf{Y}$ . Using this  $\mathbf{Y}$  we define  $Z_{t_0}^{(M)}$  by

$$Z_{t_0}^{(M)} := \inf_{\mathbf{Y}} Z(\mathbf{Y}) = \inf_{x_{t_0}} Z(\mathbf{Y}). \quad (3.1)$$

Let  $S^k$  be an arbitrary  $k$ -dimensional complex linear subspace of  $\mathbb{C}^n$ . By the Courant-Fischer representation theorem (look up Theorem 7.7 of Zhang (1999) for the Courant-Fischer representation theorem),

$$\begin{aligned} Z(\mathbf{X}) &= \frac{1}{N} \max_{S^k} \min_{\mathbf{v} \in S^k, \mathbf{v}^* \mathbf{v} = 1} \mathbf{v}^* \mathbf{X} \mathbf{X}^* \mathbf{v} \\ &= \frac{1}{N} \max_{S^k} \min_{\mathbf{v} \in S^k, \mathbf{v}^* \mathbf{v} = 1} \mathbf{v}^* \left( \sum_{t=1}^N X_t X_t^* \right) \mathbf{v} \\ &= \frac{1}{N} \max_{S^k} \min_{\mathbf{v} \in S^k, \mathbf{v}^* \mathbf{v} = 1} \mathbf{v}^* \left( \sum_{t=1}^N Y_t Y_t^* + X_{t_0} X_{t_0}^* - x_{t_0} x_{t_0}^* \right) \mathbf{v} \\ &= \frac{1}{N} \max_{S^k} \min_{\mathbf{v} \in S^k, \mathbf{v}^* \mathbf{v} = 1} \mathbf{v}^* \left( \sum_{t=1}^N Y_t Y_t^* \right) \mathbf{v} + \mathbf{v}^* (X_{t_0} X_{t_0}^* - x_{t_0} x_{t_0}^*) \mathbf{v} \\ &\leq \frac{1}{N} \max_{S^k} \min_{\mathbf{v} \in S^k, \mathbf{v}^* \mathbf{v} = 1} \mathbf{v}^* \left( \sum_{t=1}^N Y_t Y_t^* \right) \mathbf{v} + \max_{\mathbf{u} \in \mathbb{C}^n, \mathbf{u}^* \mathbf{u} = 1} \mathbf{u}^* (X_{t_0} X_{t_0}^* - x_{t_0} x_{t_0}^*) \mathbf{u} \\ &= Z(\mathbf{Y}) + \frac{1}{N} \max_{\mathbf{v} \in \mathbb{C}^n, \mathbf{v}^* \mathbf{v} = 1} \mathbf{v}^* (X_{t_0} X_{t_0}^* - x_{t_0} x_{t_0}^*) \mathbf{v}. \end{aligned}$$

Since  $|X_{lt_0}| \leq 1$  and since  $\mathbf{v}^* \mathbf{v} = 1$ , we have

$$\begin{aligned}
Z(\mathbf{X}) - Z(\mathbf{Y}) &\leq \frac{1}{N} \max_{\mathbf{v} \in \mathbb{C}^n, \mathbf{v}^* \mathbf{v} = 1} \mathbf{v}^* (X_{t_0} X_{t_0}^* - x_{t_0} x_{t_0}^*) \mathbf{v} \\
&= \frac{1}{N} \max_{\mathbf{v} \in \mathbb{C}^n, \mathbf{v}^* \mathbf{v} = 1} (\overline{X_{t_0}^* \mathbf{v}}) (X_{t_0}^* \mathbf{v}) - (\overline{x_{t_0}^* \mathbf{v}}) (x_{t_0}^* \mathbf{v}) \\
&\leq \frac{1}{N} \max_{\mathbf{v} \in \mathbb{C}^n, \mathbf{v}^* \mathbf{v} = 1} \left| \sum_{l=1}^n \overline{X_{lt_0}} v_l \right|^2 \\
&\leq \frac{1}{N} \max_{\mathbf{v} \in \mathbb{C}^n, \mathbf{v}^* \mathbf{v} = 1} \left( \sum_{l=1}^n |X_{lt_0}|^2 \right) \left( \sum_{l=1}^n |v_l|^2 \right) \\
&\leq \frac{n}{N}.
\end{aligned}$$

Take the infimum over  $x_{t_0}$ . Then, by the choice of  $Z_{t_0}^{(M)}$  given in (3.1) we have

$$0 \leq Z - Z_{t_0} \leq \frac{n}{N}.$$

So,

$$\Delta_M^2 := \sum_{t_0=1}^N (Z - Z_{t_0})^2 \leq \frac{n^2}{N}. \quad (3.2)$$

By (3.2) and by Theorem 1 (i) we have the right tail bound for  $Z = \lambda_k$ .

Now, we consider the left tail. When we choose  $Z_{t_0}$ , instead of taking the infimum this time we take the supremum. Define  $Z_{t_0}^{(L)}$  by

$$Z_{t_0}^{(L)} := \sup_{\mathbf{Y}} Z(\mathbf{Y}) = \sup_{x_{t_0}} Z(\mathbf{Y}). \quad (3.3)$$

Then, by the Courant-Fischer representation theorem we have

$$Z(\mathbf{Y}) - Z(\mathbf{X}) \leq \frac{n}{N}.$$

Take the supremum over  $x_{t_0}$ . Then, by the choice of  $Z_{t_0}^{(L)}$  given in (3.3) we have

$$0 \leq Z_{t_0}^{(L)} - Z \leq \frac{n}{N}.$$

So,

$$\Delta_L^2 \leq \frac{n^2}{N}. \quad (3.4)$$

By (3.4) and by Theorem 1 (ii), we have the left tail bound for  $Z = \lambda_k$ . ■



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